Power-law random walks

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In this paper, random walks with independent steps distributed according to a Q-power-law probability distribution function with $Q=1/(1-q)$ are studied. In the case $q>1$, we show that (i) a stochastic representation of the location of the walk after n steps can be explicitly given (for both finite and infinite variance) and (ii) a clear connection with the superstatistics framework can be established (including the anomalous diffusion case). In the case $q<1$, we prove that this random walk can be considered as the projection of an isotropic random walk, i.e., a random walk with fixed length steps and uniformly distributed directions. These results provide a natural extension of (i) the usual Gaussian framework and (ii) the infinite-covariance case of the superstatistics treatments.

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I. INTRODUCTION

The name "random walk" was originally proposed by Pearson [1] with reference to a simple model to describe mosquito infestation in a forest, although previous important work in related subjects had already been published by Lord Rayleigh. Around the same time, the theory of random walks was also developed by Bachelier in a remarkable doctoral thesis [1]. He proposed the random walk as the fundamental model for financial time series, long before this idea became the basis for modern theoretical finance. He also made the connection between discrete random walks and the continuous diffusion (or heat) equation, which is a major scientific theme. Around the same time as Pearson's work, Einstein also published his seminal paper on Brownian motion (normal diffusion), a random walk driven by collisions with gas molecules. Similar theoretical ideas were also published independently by Smoluchowski [1].

The statistical properties of random walks tend toward universal distributions after large numbers of independent steps. In the case of the concomitant probability distribution function (PDF) for the final position, the result for isotropic random walks is a multidimensional generalization of the central limit theorem (CLT) for sums of independent, identically distributed (IID) random variables. When the assumptions of the central limit theorem break down, random walks can exhibit rather different behavior from that of normal diffusion. For instance, the limiting distribution for the position of a Brownian particle may not be Gaussian. In particular, power-law distributions become of paramount importance in such a context. One way to violate the CLT with IID displacements is via "heavy-tailed" probability distributions, which assign sufficient probability to very large steps so that the variance is infinite. In this context one speaks of anomalous diffusion (AD). In an AD scenario power-law probability distributions and power-law entropies become ubiquitous. The associated literature is really vast: see, for instance, Refs. [2–6] and references therein.

In this paper we reconsider the distribution of a random walk whose independent steps follow a power-law distribution of the q Gaussian type with exponent equal to $1/(1-q)$; $q\in\mathbb{R}$. In the case $q>1$ we show that a stochastic representation of the point reached after n steps of the walk can be expressed explicitly for all n so that the superstatistics framework holds even in the anomalous diffusion case. In the case $q<1$, we show that the q Gaussian random walk can be interpreted as a projection of an isotropic random walk, i.e., a random walk with fixed length steps and uniformly distributed directions.

II. THE CASE $q>1$

A random vector $X\in\mathbb{R}^p$ is of the q Gaussian kind if its probability density is

$$f_X(X)=Z_q^{-1}(1+X^t\Lambda^{-1}X)^{1/(1-q)} \tag{2.1}$$

where the number of degrees of freedom m, dimension p, and nonextensivity parameter q are related as

$$m=\frac{2}{q-1}-p. \tag{2.2}$$

This distribution has finite covariance matrix $K=EXX^t$ provided that $m>2$ or equivalently $q<(p+4)/(p+2)$. In that case, the covariance matrix is related to the scaling matrix Λ in the fashion $\Lambda=(m-2)K$. Moreover, the partition function Z_q reads

$$Z_q=\left(\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{p}{2}\right)|\pi\Lambda|^{1/2}}\right)^{-1}.$$

Note that the usual Gaussian distribution corresponds to the limit case $q\to1^+$. We recall [7–9] that the random vector X

can be expressed as the Gaussian scale mixture[1]

$$X = \frac{\Lambda^{1/2} G}{\sqrt{a}} \tag{2.3}$$

where G is a p-variate, unit-covariance Gaussian random vector and a is a random variable independent of G that follows a χ^2 distribution with m degrees of freedom. Representation (2.3) reflects exactly the notion of superstatistics as introduced by Beck and Cohen [10–16]: a q Gaussian random system with $q>1$ can be interpreted as a Gaussian system submitted to multiplicative fluctuations following an inverse χ distribution.

A. The finite-covariance case

In the context of a random walk, we are interested in the distribution of the normalized random vector

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \tag{2.4}$$

where the random vectors $X_i \in \mathbb{R}^p$ are independent and q Gaussian distributed according to (2.1), each with m degrees of freedom and covariance matrix K. The random vector Z_n can be characterized by the following theorem.

Theorem 1. A stochastic representation of random vector Z_n is

$$Z_n \doteq \alpha_{m,n} \left(\sum_{i=1}^{n} \frac{1}{\nu_i} \right)^{1/2} X \tag{2.5}$$

where the p-variate vector X is q Gaussian distributed with nm degrees of freedom and covariance matrix K, where the constant quantity $\alpha_{m,n} = (1/n)\sqrt{(m-2)/(m-2/n)}$, while $\{\nu_1, \ldots, \nu_n\}$ are Dirichlet distributed[2] with parameters $m_1 = \cdots = m_n = m$ and independent of vector X.

Proof. We follow here the proof given in [18]: a linear combination of Gaussian scale mixtures is itself a Gaussian scale mixture since[3]

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\Lambda^{1/2} G_i}{\sqrt{a_i}} \doteq \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \frac{1}{a_i}} \Lambda^{1/2} G$$

$$= \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \frac{\sum_{j=1}^{n} a_j}{a_i}} \frac{\Lambda^{1/2} G}{\sqrt{\sum_{j=1}^{n} a_j}} \tag{2.6}$$

where G is a p-variate Gaussian vector with unit covariance.

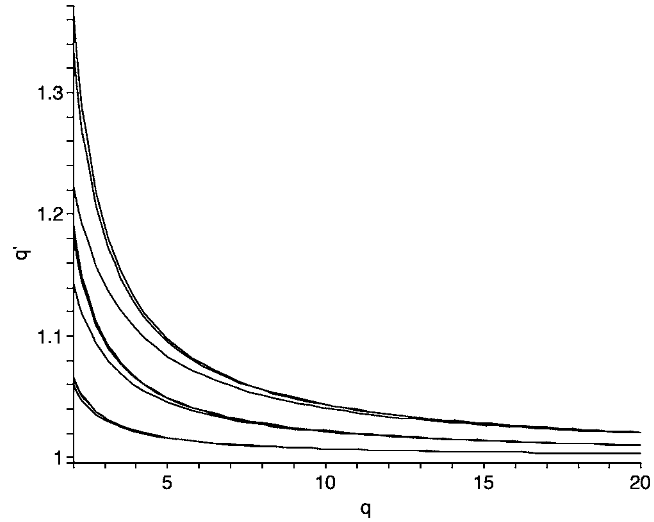

FIG. 1. q' as a function of n for $m=5$ and $q=5,3$, and 1.5 (top to bottom) on the three top curves; for $m=10$ and $q=5,3$, and 1.5 (top to bottom) on the three middle curves; for $m=30$ and $q=5,3$, and 1.5 (top to bottom) on the three lowest curves.

Now we remark that random variables a_i are χ^2 distributed with m degrees of freedom; thus, by Luckacs' result [17], each random variable $\nu_i = \frac{a_i}{\sum_{j=1}^{n} a_j}$ is independent of $\sum_{j=1}^{n} a_j$. Moreover, equality (2.1) shows that the random variables $\{\nu_i\}_{1 \le i \le n-1}$ are Dirichlet distributed. Finally, since $\sum_{j=1}^{n} a_j$ is χ^2 distributed with mn degrees of freedom, we deduce that $(\sqrt{nm-2}/\sqrt{m-2})\Lambda^{1/2} G / \sqrt{\sum_{j=1}^{n} a_j}$ is a q Gaussian vector with covariance matrix K and mn degrees of freedom. ■

An alternate proof that uses the more conventional Fourier transform tool is given in the Appendix in the case $n=2$; unfortunately, this purely analytical proof involves, even in this simple case, a complicated integral formula whose extension to an arbitrary value of n is, to our best knowledge, not available.

A striking result is thus obtained: at its nth step, a q Gaussian random walk with $q>1$ is a scale mixture of a q Gaussian vector. We note that this property holds true for any random walk with independent steps following a Gaussian scale mixture. The fact that this property extends to q Gaussian distributions is indeed remarkable. As described in the preceding proof, this special behavior is a consequence of a famous result by Lukacs [17] about the Gamma distributions, which are precisely the ones that rule the fluctuations described by the superstatistics theory [10–16].

Moreover, the nonextensivity parameter q' of vector X in (2.5) is related to the parameter q of each step as

$$q' = 1 + \frac{2(q-1)}{2 + m(n-1)(q-1)}. \tag{2.7}$$

We note that the dimension p of the random walk does not appear in this formula.

The curves in Fig. 1 represent q' as a function of n for (i) $m=5$ and $q=5,3$, and 1.5 (top to bottom) on the three top curves, (ii) $m=10$ and $q=5,3$, and 1.5 (top to bottom) on the

[1] A random vector U is a Gaussian scale mixture if $U = \sqrt{b}G$ where G is a Gaussian vector and b is a random positive variable *independent* of G.

[2] Vector $(v_1, \ldots, v_n)$ has a Dirichlet distribution with parameters $(\alpha_1, \ldots, \alpha_n)$ if its distribution has density $f(v_1, \ldots, v_n) = [\prod_{i=1}^{n} \Gamma(\alpha_i) / \Gamma(\sum_{i=1}^{n} \alpha_i)] v_1^{\alpha_1 - 1} \cdots v_n^{\alpha_n - 1}$ over the $(n-1)$-dimensional simplex $\sum_{i=1}^{n} v_i = 1$, $v_i \ge 0$.

[3] $\doteq$ denotes equality in distribution; $\Rightarrow$ denotes weak convergence.

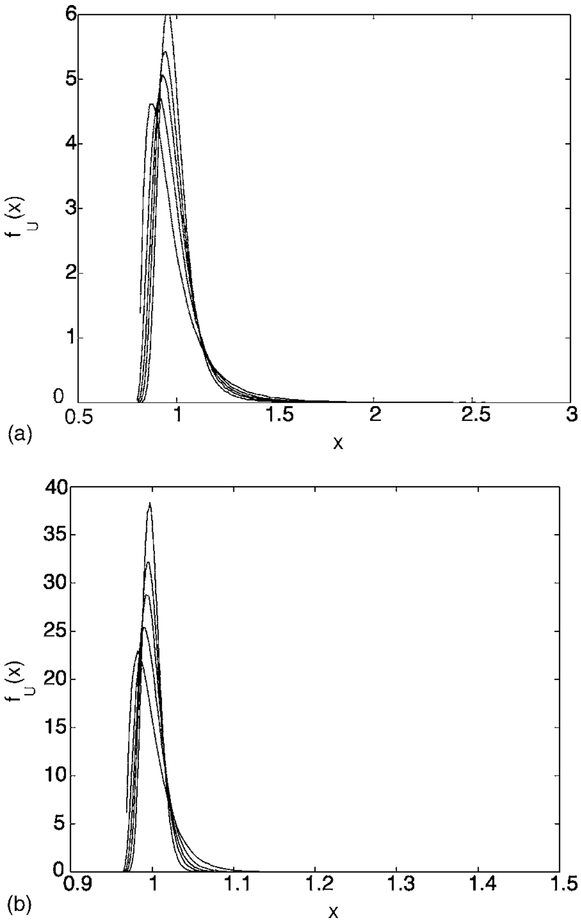

FIG. 2. Estimated PDF of $U_{5,n}$ (left) and $U_{25,n}$ (right) with $n=5,10,15,20$, and 30 (bottom to top).

three middle curves, and (iii) $m=30$ and $q=5,3$, and 1.5 (top to bottom) on the three lowest curves.

These curves confirm the three following results.

(1) Since the variance is finite, the central limit theorem applies and Z_n converges to a Gaussian vector with covariance matrix K, and thus q' converges to 1.

(2) The convergence to a Gaussian vector is all the faster since the number of degrees of freedom m is large—or, equivalently, since the independent steps X_i are closer to Gaussian steps.

(3) For a large enough value of m, the convergence process of q' to 1 is relatively insensitive to the value of q.

Unfortunately, the probability density for the scaling random variable

$$U_{m,n} = \alpha_{m,n}\left(\sum_{i=1}^{n}\frac{1}{\nu_i}\right)^{1/2} \tag{2.8}$$

cannot be explicitly given. Figure 2 depict an estimation of the probability distribution function for $U_{m,n}$ after $n=5,10,15,20$, and 30 steps of the random walk in the cases $m=5$ and 25. Note that different scales have been employed. These figures clearly exhibit the convergence of the random variable $U_{m,n}$ to the deterministic unit constant, as required by the central limit theorem.

B. The infinite-covariance case: Lévy flights

Let us assume now $m<2$ so that each of the steps of the random walk has infinite covariance. Let us consider the unnormalized random walk

$$Z_n = \sum_{i=1}^{n} X_i. \tag{2.9}$$

The distribution of any component $X_i^{(k)}$ of vector X_i,

$$f_x(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\sqrt{\pi\Lambda_{k,k}}}\left(1+\frac{x^2}{\Lambda_{k,k}}\right)^{-(m+1)/2},$$

behaves as

$$f_x(x) \sim |x|^{-m-1}, \quad |x| \to +\infty, \tag{2.10}$$

so that the number of degrees of freedom m, for $m\leq 2$, coincides with the Lévy index of the q Gaussian distribution. Now, by direct application of the Lévy-Gnedenko central limit theorem [19–21] one immediately realizes that

$$\frac{1}{n^{1/m}}\Lambda^{-1/2}Z_n \Rightarrow S_m, \tag{2.11}$$

where S_m denotes a vector, each component of which follows a symmetric α-stable distribution with Lévy index m.

A quite interesting result worth quoting at this point is that, although the involved variables have infinite covariance, the superstatistics principle still applies in the following fashion,

Theorem 2. For all n, the normalized random walk $n^{-1/m}Z_n$ is distributed as a Gaussian scale mixture. Further, the distribution of the mixing variable converges, as $n\to+\infty$, to a stable distribution with Lévy index $\alpha=m/2$.

Proof. We use the first part of the proof of Theorem 1:

$$Z_n = \sum_{i=1}^{n}\frac{\Lambda^{1/2}G_i}{\sqrt{a_i}} \doteq \sqrt{\sum_{i=1}^{n}\frac{1}{a_i}}\Lambda^{1/2}G. \tag{2.12}$$

One can easily check that each random variable $1/a_i$ has Lévy index $m/2$. Now, the Lévy-Gnedenko theorem yields

$$\frac{1}{n^{2/m}}\sum_{i=1}^{n}\frac{1}{a_i} \Rightarrow S_{m/2},$$

so that

$$\frac{1}{n^{1/m}}Z_n \Rightarrow \sqrt{S_{m/2}}\Lambda^{1/2}G.$$

■

Note that this result is coherent with the representation (2.11) as given by the Lévy-Gnedenko theorem. Indeed, according to a classical result about stable random variables [22], if S_α and $S_{\alpha'}$ denote two independent stable random variables with respective indices α and α', then

$$S_\alpha S_{\alpha'}^{1/\alpha} \doteq S_{\alpha''}$$

with $\alpha'' = \alpha\alpha'$. In the above situation, this result applies componentwise with $\alpha=2$ and $\alpha'=m/2$.

Note also that the set of Lévy stable distributions and the set of q Gaussian distributions do not coincide, although Lévy stable distributions are Gaussian scale mixtures [23]; in fact, these sets have only two common elements: the Gaussian distribution (with nonextensivity index $q=1$ and Lévy index $\alpha=2$) and the Cauchy distribution (with nonextensivity index $q=0$ and Lévy index $\alpha=1$). However, q Gaussian and Lévy distributions share the same heavy-tail asymptotic behavior: a q Gaussian distribution with parameter $q>1$ shows the same asymptotic behavior $f_X(X) \sim |X|^{2/(1-q)}$ as a Lévy distribution with parameter $\alpha=(3-q)/(q-1)$.

III. THE CASE $q<1$

The p-variate q Gaussian distribution in the case $q<1$ is written explicitly as

$$f_X(X) = Z_q^{-1}(1 - X^t\Lambda^{-1}X)_+^{1/(1-q)} \tag{3.1}$$

with notation $(x)_+=\max(x,0)$. The covariance matrix of vector X is finite and is written $K=EXX^t=d^{-1}\Lambda$ with $d=p+2(2-q)/(1-q)$. Moreover, the partition function Z_q is written

$$Z_q = \left(\frac{\Gamma\left(\frac{2-q}{1-q} + \frac{p}{2}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)|\pi\Lambda|^{1/2}} \right)^{-1}. \tag{3.2}$$

A stochastic representation of a vector X following this distribution is

$$X \doteq \frac{\Lambda^{1/2}G}{\sqrt{G^tG + b}}, \tag{3.3}$$

where the random variable b is χ^2 distributed with $2(2-q)/(1-q)$ degrees of freedom and independent of the p-variate, unit-variance Gaussian vector G. Let us consider now the random walk

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, \tag{3.4}$$

where the random vectors X_i are independent and follow distribution (3.1).

Although, contrarily to the case $q>1$, no explicit stochastic representation can be provided for (3.4), this kind of random walk can be given an interesting interpretation, as follows.

Theorem 3. If Y_n is an isotropic d-dimensional random walk [1]

$$Y_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n T_i$$

where $T_i \in \mathbb{R}^d$ are independent random vectors with unit length $\|T_i\|=1$ and uniformly distributed direction[4] and if

$$2\frac{2-q}{1-q} \in \mathbb{N} \tag{3.5}$$

then $\Lambda^{-1/2}Z_n$ is the p-dimensional marginal of Y_n with

$$p = d - 2\frac{2-q}{1-q}. \tag{3.6}$$

Proof. A vector T_i uniformly distributed on the sphere $\mathcal{S}_d$ has stochastic representation $T_i \doteq \tilde{G}/\|\tilde{G}\|$ where $\tilde{G} \in \mathbb{R}^d$ is a Gaussian random vector with unit covariance. Thus stochastic representation (3.3) shows that $\Lambda^{-1/2}X$ is the p-variate marginal of T_i (see [9]). ■

In a physical context, this result can be interpreted as follows: assume that we observe a p-dimensional q random walk whose nonextensivity parameter $q<1$ verifies condition (3.5); then a reasonable hypothesis is that one observes only a part (some components) of a higher-dimensional random walk, namely, a d-variate isotropic random walk with d defined as in (3.6).

Another useful result about the q Gaussian random walk Z_n, as defined by (3.4) with $q<1$, is given by the following theorem.

Theorem 4. If (i) $Z_n \in \mathbb{R}^p$ is a q Gaussian random walk with $q<1$ and (ii) $\{a_i\}_{1\le i\le n}$ are independent random variables [independent of all X_i's for $1\le i\le n$ in Eq. (3.4)] that follow a χ distribution with d degrees of freedom such that

$$d = p + 2\frac{2-q}{1-q}, \tag{3.7}$$

then the random walk

$$\tilde{Z}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n a_iX_i \tag{3.8}$$

is a Gaussian random walk with independent steps, each with covariance Λ.

Proof. The fact that each step a_iX_i is a Gaussian vector is proved in [7]. The covariance matrix of a_iX_i is easily computed as

$$E(a_iX_i)(a_iX_i^t) = Ea_i^2EX_iX_i^t = d\frac{1}{d}\Lambda = \Lambda \tag{3.9}$$

and the independence of the steps results from the assumptions. ■

IV. CONCLUSIONS

We have proved here some results for random walks governed by distributions of the power-law type, summarized as follows.

[4]In other words, each T_i is uniformly distributed on the unit sphere $\mathcal{S}_p=\{X\in\mathbb{R}^p \mid \|X\|=1\}$.

(1) In the case $q > 1$ (Sec. II) we saw that a stochastic representation of the point reached after n steps of the walk can be expressed explicitly for all n.

(2) Moreover, Theorem 2 allows one to highlight the fact that, even in the Lévy (infinite covariance) case, the superstatistics framework still remains valid, a rather remarkable result.

(3) In the case $q < 1$ (Sec. III), we ascertained that the random walk can be interpreted as a projection of an isotropic random walk, i.e., a random walk with fixed length steps and uniformly distributed directions.

(4) Moreover, Theorem 4 shows that a q Gaussian random walk with $q < 1$, each step of which is subjected to independent, multiplicative χ-distributed fluctuations, is exactly a Gaussian random walk, a fact that can qualify as *dual superstatistics*.

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APPENDIX: THE FOURIER PROOF OF THEOREM 1

A simple analytical proof of Theorem 1 based on characteristic functions can be provided in the case $n=2$. We first remark that, up to a left multiplication of (2.5) by matrix K^{-1}, it may be assumed without loss of generality that $K = I_p$. The characteristic function $\phi_X(u)$ of each X_i is

$$\phi_X(u) = \frac{2^{1-m/2}}{\Gamma(m/2)}|u(m-2)|^{m/2}K_{m/2}[(m-2)|u|]$$

so that the characteristic function of $Z_2 = (1/\sqrt{2})(X_1 + X_2)$ is

$$\phi_Z(u) = \phi_X^2\left(\frac{u}{\sqrt{2}}\right) = \frac{2^{2-m}}{\Gamma^2(m/2)}\left|\frac{u(m-2)}{2}\right|^m K_{m/2}^2\left(\left|\frac{u(m-2)}{2}\right|\right).$$

It can be shown easily that the distribution of $V = \sqrt{1/\nu_1 + 1/\nu_2}$ is $f_V(V) = \frac{4}{B(m/2,m/2)}\frac{v^{-m}}{\sqrt{v^2-4}}$, $v \geq 2$, so that the characteristic function of $W = VX$, where X is q Gaussian distributed with $2m$ degrees of freedom, is

$$\phi_W(u) = \int_2^{+\infty} f_V(v)\phi_X(uv)dv$$
$$= \frac{2^{3-m}|u(2m-2)|^m}{\Gamma^2(m/2)}\int_2^{+\infty}\frac{1}{\sqrt{v^2-4}}K_m[|uv|(2m-2)]dv.$$

The last integral can be expressed using [[24], 6.592.8] as

$$\int_2^{+\infty}\frac{1}{\sqrt{v^2-4}}K_m[|uv|(2m-2)]dv = \frac{1}{2}K_{m/2}^2[|u|(2m-2)]$$

so that the characteristic function of $\tilde{Z}_2 = \alpha_{m,2}\sqrt{1/\nu_1 + 1/\nu_2}X$ is

$$\phi_{\tilde{Z}}(u) = \frac{2^{2-m}}{\Gamma^2(m/2)}\left|u\frac{m-2}{2}\right|^m K_{m/2}^2\left(\frac{m-2}{2}|u|\right) = \phi_Z(u)$$

so that Z_2 and $\tilde{Z}_2$ have the same distribution. Unfortunately, we are not aware of any version of formula [[24], 6.592.8] that would allow us to extend this proof to an arbitrary value of n.

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